Quantum breaking time near classical equilibrium points

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By using numerical and semiclassical methods, we evaluate the quantum breaking, or Ehrenfest time for a wave packet localized around classical equilibrium points of autonomous one-dimensional systems with polynomial potentials. We find that the Ehrenfest time diverges logarithmically with the inverse of the Planck constant whenever the equilibrium point is exponentially unstable. For stable equilibrium points, we have a power law divergence with exponent determined by the degree of the potential near the equilibrium point.

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The question of estimating how long classical and quantum evolutions stay close is one of the main problems of semiclassical analysis. The evolution of a quantum observable can follow that of the corresponding classical one up to a finite time, the so called quantum breaking, or Ehrenfest, time. As initially conjectured in [1, 2] and rigorously proved in [3, 4, 5], whenever the classical flow is chaotic, the Ehrenfest time diverges logarithmically in $\hbar$. This result is easily understood. Starting from an initial value $\Delta(\hbar) \sim \hbar/I$, where I is a characteristic action of the system, the difference between a classical flow, with Lyapunov exponent $\lambda > 0$, and the corresponding quasi-periodic quantum flow increases as $\Delta(\hbar)\exp(\lambda t)$. The two flows depart at $t \sim \lambda^{-1}\log(I/\hbar)$. The situation is different for a regular classical flow. In this case, starting from the work [6], it was suggested in [7] that the Ehrenfest time grows algebraically as $\hbar^{-\delta}$. The determination of the value of δ and its possible universal nature is still an open problem. See [8] and references therein for recent results.

The $\hbar$-scaling of the Ehrenfest time is usually investigated for classical flows which are completely chaotic or regular. However, it is interesting to study the quantum-classical correspondence in systems having isolated unstable orbits embedded in a completely regular phase-space. The simplest example is given by the ubiquitous double-well system defined by the Hamiltonian $H(p,q) = \frac{p^2}{2} - \frac{q^2}{2} + \frac{q^4}{4}$. For this system, there is only one unstable periodic orbit, namely that associated to the equilibrium point $(p_0, q_0) = (0, 0)$, with positive Lyapunov exponent $\lambda = 1$. Is it possible to have a logarithmic Ehrenfest time in proximity of an isolated exponentially unstable point like (p_0, q_0)?

The usual way of studying the Ehrenfest time consists in comparing the evolution of classical observables with the quantum expectation value of the corresponding operators, either in the coherent state representation [3], or in the framework of Weyl quantization [4]. In the present case, we follow a simpler approach based on the analysis of the quantum spectrum. We know that on going towards the classical equilibrium point (p_0, q_0) the period of motion diverges, so that the evolution of a phase-space distribution function localized around this point must show a continuous frequency distribution around $\nu = 0$. On the other hand, in the quantum case, due to the discrete nature of the spectrum, the frequency distribution is characterized by a gap between zero and a minimal frequency. We call this minimal frequency the Ehrenfest frequency, ν_E. In fact, its inverse, ν_E^{-1}, is an upper bound to the time at which the quantum-classical correspondence of the evolution of any observable breaks down. We estimate the Ehrenfest time as ν_E^{-1}.

By using numerical and semiclassical methods, we study the behavior of $\nu_E(\hbar)$ around classical equilibrium points, both stable and unstable, for several autonomous one-dimensional systems. We find that $\nu_E^{-1}(\hbar)$ diverges logarithmically for $\hbar \to 0$ whenever the equilibrium point is exponentially unstable. In all the other cases, the Ehrenfest time follows a power law with exponent related to the degree of the potential near the equilibrium point.

In the following, we consider systems described by the Hamiltonians

$$H(p,q) = \frac{p^2}{2m} + A\frac{q^{2\alpha}}{2\alpha} + B\frac{q^{2\beta}}{2\beta}, \qquad (1)$$

with $A \leq 0$, $B > 0$ and $\beta > \alpha \geq 1$. By properly rescaling position, momentum and time, we can always reduce to the case $B = 1$, $m = 1$ and either $A = 0$ or $A = -1$ [16]. For $A = 0$, we have single-well systems with a classical stable equilibrium point $(p_0, q_0) = (0, 0)$ at energy $\varepsilon = 0$. A more interesting situation occurs for $A = -1$. In this case the systems are double-well oscillators and the classical equilibrium point $(p_0, q_0) = (0, 0)$ at energy $\varepsilon = 0$ is unstable. In the particular case $\alpha = 1$, the equilibrium point is exponentially unstable. In both cases, $A = 0$ or $A = -1$, the periodic orbits near the equilibrium point at $\varepsilon = 0$ have a period which diverges for $\varepsilon \to 0$.

On the quantum mechanical side, in order to represent a state localized near the classical equilibrium point

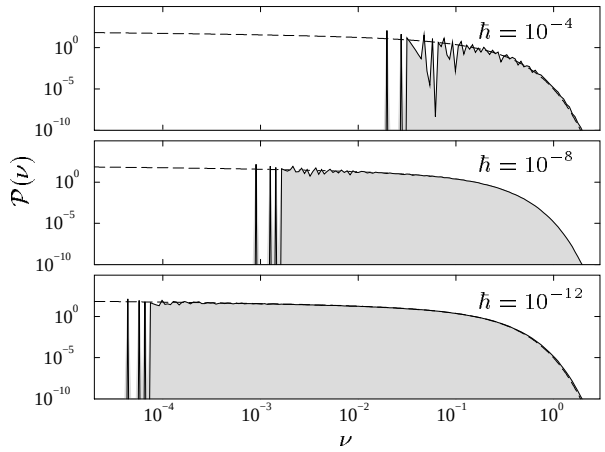

FIG. 1: Fourier transform of the survival probability $\mathcal{P}(\nu)$ for different values of $\hbar$ in the case $A = 0$, $\beta = 2$. The dashed line is the $\hbar \to 0$ limit distribution given by Eq. (14).

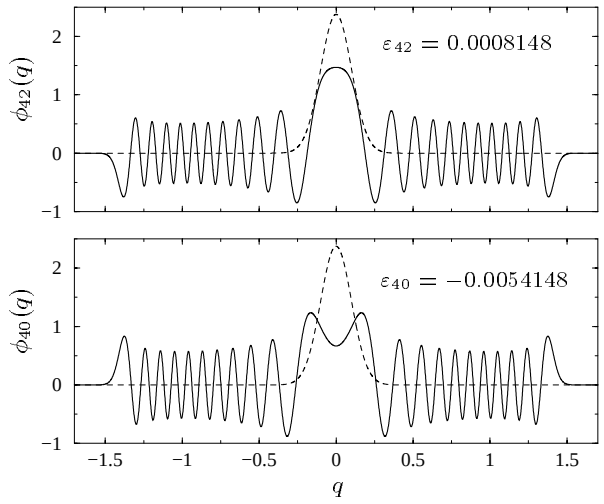

FIG. 2: Eigenfunctions ϕ_{40} and ϕ_{42} corresponding to the minimal frequency ν_E in the double-well case $\alpha = 1$, $\beta = 2$ for $\hbar = 10^{-2}$. The dashed curve is the initial wavefunction (2) with $(p_0, q_0) = (0, 0)$.

(p_0, q_0) we consider the following initial wavefunction

$$\langle q|\psi(0)\rangle = \frac{1}{(\pi\hbar)^{\frac{1}{4}}} \exp\left[-\frac{(q-q_0)^2}{2\hbar}\right] \exp\left(i\frac{p_0 q}{\hbar}\right). \quad (2)$$

The associated Wigner function,

$$W_\psi(p,q) = \frac{1}{\pi\hbar} \exp\left[-\frac{(p-p_0)^2}{\hbar}\right] \exp\left[-\frac{(q-q_0)^2}{\hbar}\right], \quad (3)$$

can be interpreted as a phase-space probability distribution centered around the point (p_0, q_0) and has the property

$$\lim_{\hbar\to 0} W_\psi(p,q) = \delta(p-p_0)\delta(q-q_0). \quad (4)$$

In these expressions $\hbar$ is the adimensional rescaled Planck constant, which vanishes when, for instance, the mass m of the system is taken larger and larger.

Instead of considering the evolution of a specific observable, we study the simpler survival probability

$$\mathcal{P}(t) = |\langle\psi(0)|\psi(t)\rangle|^2, \quad (5)$$

which contains the same gross dynamical information. In the basis of the eigenstates of the Hamiltonian,

$$H|\phi_n\rangle = \varepsilon_n|\phi_n\rangle, \quad n = 0, 1, 2, \ldots, \quad (6)$$

the survival probability $\mathcal{P}(t)$ can be written as

$$\mathcal{P}(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |c_n|^2|c_m|^2 \exp\left(i\nu_{nm}t\right), \quad (7)$$

where $c_k = \langle\psi(0)|\phi_k\rangle$ and $\nu_{nm} = (\varepsilon_n - \varepsilon_m)/(2\pi\hbar)$. Note that $c_k = 0$ for k odd, due to the symmetry of the system and of the initial wavefunction.

By using semiclassical and numerical techniques, we now show that the Fourier transform of the survival probability,

$$\mathcal{P}(\nu) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |c_n|^2|c_m|^2\delta(\nu - \nu_{nm}), \quad (8)$$

for sufficiently small values of $\hbar$ is characterized by a gap, large with respect to the typical level spacing, between $\nu = 0$ and a frequency which we call the Ehrenfest frequency, defined as

$$\nu_E = \min_{\substack{n\neq m \\ |c_n|^2|c_m|^2\neq 0}} \nu_{nm}. \quad (9)$$

In the simple case $A = 0$, by using standard WKB approximations, we have

$$\varepsilon_n = \left[\left(n + \frac{1}{2}\right)\hbar\delta(\beta)\right]^{\frac{2\beta}{\beta+1}}, \quad (10)$$

with

$$\delta(\beta) = \sqrt{\frac{\pi}{2}} \frac{\Gamma\left[\frac{1}{2}\left(3 + \frac{1}{\beta}\right)\right]}{\Gamma\left(1 + \frac{1}{2\beta}\right)(2\beta)^{\frac{1}{2\beta}}}, \quad (11)$$

and

$$|c_\varepsilon|^2 = \frac{2\sqrt{\pi}\,(2\beta)^{-\frac{1}{2\beta}}\,\hbar^{\frac{1}{2}}\varepsilon^{-\frac{1}{2\beta}}e^{-2\frac{\varepsilon}{\hbar}}}{\frac{\Gamma(\frac{1+\beta}{2\beta})}{\Gamma(\frac{1}{2})\Gamma(1+\frac{1}{2\beta})} + \frac{\sin\sigma(\varepsilon,\hbar;\beta)}{\sigma(\varepsilon,\hbar;\beta)}}, \quad (12)$$

with $\sigma(\varepsilon, \hbar; \beta) = 2\sqrt{2}(2\beta)^{\frac{1}{2\beta}}\hbar^{-1}\varepsilon^{\frac{\beta+1}{2\beta}}$. The behavior of $\mathcal{P}(\nu)$ obtained by using these expressions for ε_n and $|c_{\varepsilon_n}|^2$

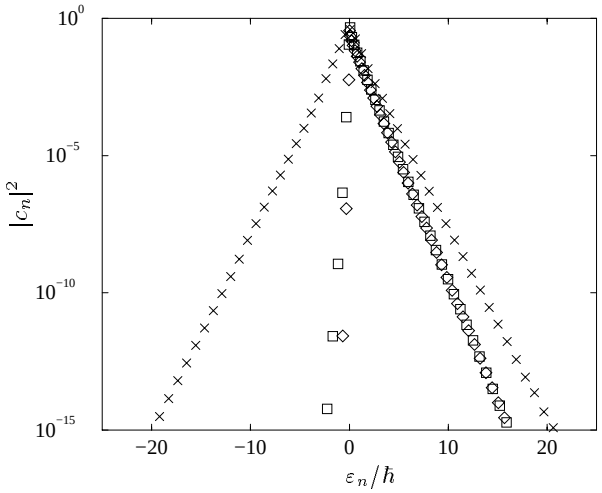

FIG. 3: Superposition coefficients $|c_n|^2$ as a function of $\varepsilon_n/\hbar$ for $\hbar = 10^{-3}$ in the double-well cases $\alpha = 1$, $\beta = 2$ ($\times$), $\alpha = 2$, $\beta = 4$ ($\Box$), and $\alpha = 3$, $\beta = 6$ ($\Diamond$).

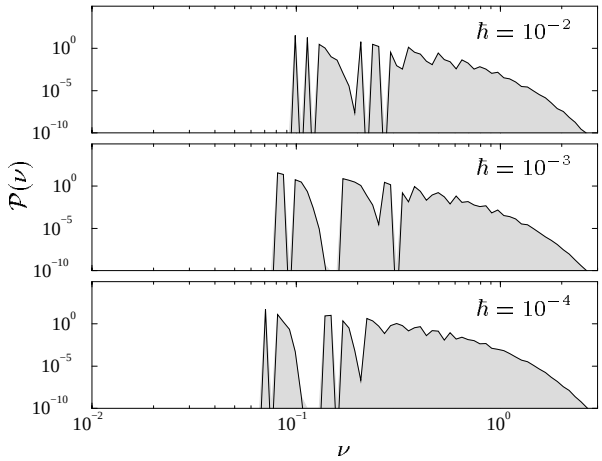

FIG. 4: Fourier transform of the survival probability $\mathcal{P}(\nu)$ for different values of $\hbar$ in the double-well case $\alpha = 1$, $\beta = 2$.

is shown in Fig. 1 in the case $\beta = 2$. We see that for $\hbar \to 0$ the frequency distribution $\mathcal{P}(\nu)$ approaches a continuous limit given by

$$\mathcal{P}_0(\nu) = \lim_{\hbar\to 0} \int \mathrm{d}\varepsilon_1 \mathrm{d}\varepsilon_2 \; p(\varepsilon_1) p(\varepsilon_2) \delta\left(\nu - \frac{\varepsilon_1 - \varepsilon_2}{2\pi\hbar}\right), \quad (13)$$

where $p(\varepsilon) = |c_\varepsilon|^2 \frac{\mathrm{d}n}{\mathrm{d}\varepsilon}$ and $n(\varepsilon)$ is obtained by inverting $\varepsilon = \varepsilon_n$. By using (10) and (12), we find

$$\mathcal{P}_0(\nu) = 4K_0(4\pi|\nu|), \quad (14)$$

where K_0 is the Bessel function of zero-th order. Figure 1 also shows the presence of the gap at $\nu = 0$ and its shrinking as $\hbar \to 0$. Since the level spacing $\varepsilon_{n+1} - \varepsilon_n$ increases by increasing n, the Ehrenfest frequency (9) turns out to be $\nu_E = (\varepsilon_2 - \varepsilon_0)/(2\pi\hbar)$. According to (10), its inverse diverges as

$$\nu_E^{-1} \sim \hbar^{\frac{1-\beta}{1+\beta}}. \quad (15)$$

We now consider double-well systems, i. e., the case $A = -1$. For these systems, the standard WKB approximation fails near the unstable equilibrium point at energy $\varepsilon = 0$. Only in the particular case $\alpha = 1$, a regularized semiclassical approximation has been developed [9, 10, 11] and the quantization condition for the energy levels reads

$$\frac{1}{\sqrt{1 + \exp\frac{2\pi\varepsilon}{\hbar}}} = \cos(\phi(\varepsilon, \hbar)), \quad (16)$$

where

$$\phi(\varepsilon, \hbar) = \frac{4}{3\hbar} - \frac{\varepsilon}{\hbar}\log\frac{\hbar}{16} - \arg\Gamma\left(\frac{1}{2} + i\frac{\varepsilon}{\hbar}\right) - \pi. \quad (17)$$

For the associated eigenfunctions only microlocal expressions are available [10] which do not allow for a direct determination of the superposition coefficients $|c_\varepsilon|^2$. For this reason, in all cases $\alpha \geq 1$ we determine numerically the eigenvalues and eigenfunctions of the system. With standard numerical techniques, this represents an unsurmountable task since the interesting eigenstates, namely those close to energy $\varepsilon = 0$, have a quantum number n which diverges quickly for $\hbar \to 0$. We bypass the problem by using the algorithm [12] which allows to evaluate selected eigenstates having a very large number of nodes. In Fig. 2 we show, as an example, the couple of even eigenfunctions with energy closest to $\varepsilon = 0$ in the double-well case $\alpha = 1$, $\beta = 2$ evaluated for $\hbar = 10^{-2}$. Note that, already for this still relatively large value of $\hbar$, the corresponding quantum number is $n \sim 40$. In our numerical calculations we go beyond $n \sim 10^4$.

In Fig. 3 we show the superposition coefficients evaluated for different double-well systems for $\hbar = 10^{-3}$. We see that $|c_n|^2$ decreases exponentially departing from $\varepsilon = 0$. For smaller values of $\hbar$, the superposition coefficients $|c_n|^2$ follow approximately the same exponential behavior as a function of $|\varepsilon_n|/\hbar$ and become denser and denser.

The Fourier transform of the survival probability (8) is determined by using the eigenvalues and the superposition coefficients obtained numerically. In Fig. 4 we show $\mathcal{P}(\nu)$ in the case $\alpha = 1$, $\beta = 2$ for different values of $\hbar$. As in the single-well case, at $\nu = 0$ we have a gap whose width shrinks as $\hbar \to 0$. The width of this gap, namely the Ehrenfest frequency, is yielded by a couple of even consecutive eigenvalues, close to the energy $\varepsilon = 0$ of the classical equilibrium point. This can be understood roughly in the following way. Consider the number of states, $\mathcal{N}_\varepsilon$, in the energy range $[\varepsilon - \hbar, \varepsilon + \hbar]$. The frequencies associated to the eigenvalues in this energy range are

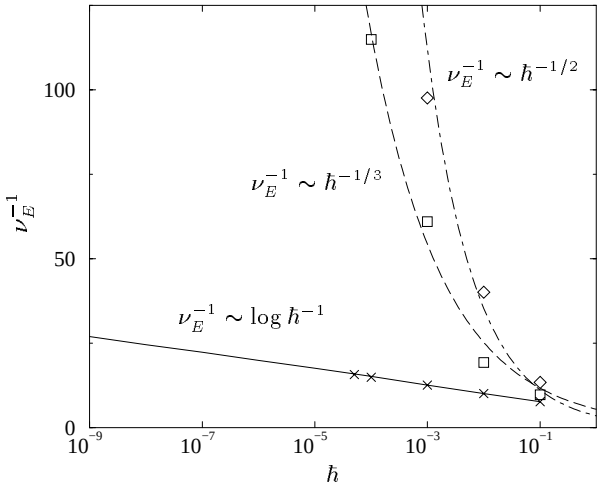

FIG. 5: Inverse of the Ehrenfest frequency, ν_E^{-1}, as a function of $\hbar$ in the double-well cases $\alpha = 1$, $\beta = 2$ ($\times$), $\alpha = 2$, $\beta = 4$ ($\Box$), and $\alpha = 3$, $\beta = 6$ ($\Diamond$). The solid line is the regularized WKB prediction based on (16-17), while the dashed and dot-dashed lines are numerical fits.

$\nu \sim \mathcal{N}_\varepsilon^{-1}$, so that, in the limit $\hbar \to 0$, ν vanishes if $\mathcal{N}_\varepsilon$ diverges. According to Weyl formula, $\mathcal{N}_\varepsilon$ is proportional to the classical phase-space volume bounded by the energy shells $H(p,q) = \varepsilon \pm \hbar$. This volume can be evaluated exactly in terms of simple functions in the single-well case and in terms of special functions for double-well systems. In all cases, we have that $\mathcal{N}_\varepsilon$ diverges when $\hbar \to 0$ only for $\varepsilon = 0$. In the double-well systems, for $\alpha = 1$, $\beta = 2$ the couple of closest eigenvalues has energies of opposite sign, as shown in Fig. 2. For $\alpha > 1$ these eigenvalues are both positive if $\hbar$ is sufficiently small.

The scaling of ν_E^{-1} with $\hbar$ is shown in Fig. 5 for different double-well systems. The plotted points are calculated using the numerically determined spectrum while the solid line represents the inverse of the Ehrenfest frequency as determined by using the quantization condition (16-17). The Ehrenfest time increases logarithmically with $\hbar^{-1}$ only in the case $\alpha = 1$, $\beta = 2$, i. e., when the equilibrium point is exponentially unstable. In all the other cases, a numerical fit suggests that

$$\nu_E^{-1} \sim \hbar^{\frac{1-\alpha}{1+\alpha}}. \qquad (18)$$

This is the same scaling law which we would obtain, as described by Eq. (15), in the case of a single-well potential $V(q) = q^{2\alpha}/(2\alpha)$. This fact can be understood in the following way. For $\hbar \to 0$, the discrete eigenvalues of the double-well above $\varepsilon = 0$ correspond to the energies of the continuous spectrum of the barrier $-q^{2\alpha}/(2\alpha)$ at which the transmission coefficient is maximum. According to WKB approximation, these resonances of the continuous spectrum in turn coincide with the energies of the bound states of the corresponding confining inverted potential.

In conclusion, we have shown that the presence of isolated exponentially unstable orbits is sufficient to break the quantum-classical correspondence at a time scale logarithmic in $\hbar^{-1}$. This feature may be relevant in all mesoscopic systems which are modeled by one-dimensional multi-well Hamiltonians [13, 14]. In these systems the Ehrenfest time behavior is related to experimentally detectable properties as the classical to quantum crossover of the shot noise [15].

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[16] For $A = 0$, the physical energy and the Planck constant are given in terms of the corresponding rescaled quantities via the substitution $\varepsilon \to \varepsilon\, m^{-\frac{\beta}{\beta-1}} B^{\frac{1}{\beta-1}} \tau^{\frac{2\beta}{\beta-1}}$, $\hbar \to \hbar\, m^{-\frac{\beta}{\beta-1}} B^{\frac{1}{\beta-1}} \tau^{\frac{\beta+1}{\beta-1}}$, where τ is an arbitrary time scale unit. In the double-well case, $A < 0$, we have $\varepsilon \to \varepsilon\, (-A)^{-\frac{\beta}{\beta-\alpha}} B^{\frac{\alpha}{\beta-\alpha}}$ and $\hbar \to \hbar\, m^{-\frac{1}{2}} (-A)^{-\frac{\beta+1}{2(\beta-\alpha)}} B^{\frac{\alpha+1}{2(\beta-\alpha)}}$.